Propagation of Weak Disturbances in a Vibrationally Relaxing Gas

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One-dimensional nonsteady flows of an inviscid relaxing gas are treated by the method of integral relations. The flows are thought of as being produced in a semi-infinite cylindrical pipe terminated by a piston, when the piston begins to move into or out of the cylinder. The velocity of the piston has acceleration discontinuities. It is assumed that the piston speed u_p is small compared with the frozen speed of sound in an undisturbed gas c_{f0}, and all the dependent variables can be described in the form of perturbation expansions in powers of a small parameter $\epsilon = u_p / c_{f0}$. An approximate semi-analytical solution for the first-order problem is obtained by the method of integral relations and sample numerical calculations are systematically carried out for a vibrationally relaxing diatomic gas. Conflicting effects of convection and vibrational relaxation on the propagation of weak nonlinear waves are investigated in detail.

I. Introduction

ONE of the simplest examples of an important class of one-dimensional nonsteady gas flow is given by the gas flow in a semi-infinite cylindrical pipe terminated by a piston, when the piston begins to move with nonzero acceleration. Although the piston problem can be solved exactly for many important cases in classical gasdynamics,[1,2] the same cannot be said in the dynamics of real gases, where various thermodynamically irreversible processes are involved. The solution of this problem for the real gas is well-known only in the linear approximation (acoustic approximation).[3-4] Several additional features emerge when nonlinearity is taken into account, and many piston problems remain to be investigated in the dynamics of real gases.

In real gases, in general, two sets of influences are present in practice, which tend to alter the acoustic waveform from that predicted by the classical linear theory. One set consists of the various thermodynamically irreversible processes (viscosity, self-diffusion, conductivity, relaxation of internal degree of freedom, chemical reaction, etc.). The other set of influences consists of the nonlinear effects. In this paper, this conflicting influence of these two sets on propagation of pressure wave is investigated in detail. The structure of a pressure wave in a vibrationally relaxing gas is studied theoretically, mainly numerically, with neglect of viscosity and other diffusive effects.

In the case of weak waves generated by the piston motion with small velocity, a simplification (the concept of bulk viscosity) has been introduced by Lighthill[5] in the equations of motion. This concept of bulk viscosity is used in conjunction with Burger's equation in the near-equilibrium region (for large time). For small time a near-frozen solution has been obtained by Jones[6] using a characteristic perturbation method. Wood and Parker[7] have studied both analytically and numerically the case of a centered rarefaction wave into an ideal relaxing gas without linearizing the flow equations.

Here we shall be concerned with weak nonlinear waves in a vibrationally relaxing gas produced by a piston with small velocity in a semi-infinite cylindrical pipe. It is important to notice that two conditions must be satisfied if the relaxation of a particular degree of freedom is to affect the wave phenomena: the relaxation time must be comparable in magnitude with the acoustic periods, and the change in energy associated with the relaxation mode must form a significant part of the total change of enthalpy of the gas. The energy stored in the vibrational mode of ordinary diatomic gases such as H_2, N_2, and O_2 is at most 10% of the total enthalpy. From the foregoing discussions it may be suggested that the vibrational relaxation alone cannot appreciably affect the wave phenomena in usual cases. This is, however, not exactly the case. For the relaxation effect of vibration as well as chemical reaction is cumulative in discussing the behavior of a pressure wave with its propagation along particular positive characteristics. This would indicate an indispensable role which may be played by the relaxation effect of vibration, however small it is locally, in discussing wave phenomena.

The problem is formulated on the characteristic space and the results are translated back into the physical space. Weak nonlinear waves are studied by the same characteristic perturbation method that was used by Chu,[8] who obtained a formal solution by the method of Laplace transforms. Here the equations for the first-order variables are integrated numerically by the method of integral relations. For several forms of piston motions with acceleration discontinuities, systematic calculations are carried out on the electronic digital computer FACOM 230 at computing center of Kyoto University. The details of wave shapes for both small time (in the nearly frozen region) and intermediate time (in the nonequilibrium region) are determined, and also their successive changes due to the vibrational relaxation and convection are investigated along the positive characteristics.

II. Basic Equations

Here we consider the disturbances in a relaxing gas, of which the molecules have only one lagging internal mode, and neglect the effects of viscosity, heat conduction, and body forces. When the characteristics, a wave tag α and a particle tag ψ, are used as the reference coordinate system, the governing equations of the one-dimensional nonsteady flow are given by Ref. 8 as

$$c_f \rho_\alpha - \rho \left(u_\alpha - \frac{t_\alpha}{t_\psi} u_\psi \right) = 0 \tag{1}$$

$$\rho c_f u_\alpha - p_\alpha + \frac{t_\alpha}{t_\psi} p_\psi = 0 \tag{2}$$

$$q_\alpha = t_\alpha Q(p.s,q) \tag{3}$$

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$$s_\alpha = \frac{t_\alpha}{T} A(p,s,q) Q(p,s,q) \tag{4}$$

$$T\mathrm{d}s = \mathrm{d}h - \frac{\mathrm{d}p}{\rho} + A\mathrm{d}q \tag{5}$$

$$h = h(p,s,q) \tag{6}$$

$$x_\alpha = u t_\alpha \tag{7}$$

$$x_\psi = (u + c_f) t_\psi \tag{8}$$

where the partial differentiations with respect to α and ψ are denoted by subscripts, t is the time, x the distance along a cylindrical pipe; ρ, u, p. q, s, T, A, and h denote the density, velocity, pressure, progress variable, entropy, temperature, affinity, and specific enthalpy, respectively. Q is assumed to be a known function of p, s, and q. The characteristic speed of this system is the frozen speed of sound c_f, which is defined by

$$c_f^2 = \left(\frac{\partial p}{\partial \rho} \right)_{s,q} \tag{9}$$

where subscripts signify the variables to be held constant during the partial differentiation. It is easily seen that once the canonical equation of state (6) is specified, ρ, T, and A are known from the Gibbs relation, Eq. (5), as the functions of p, s, and q as follows

$$\frac{1}{\rho} = \left(\frac{\partial h}{\partial p} \right)_{s,q}, \quad T = \left(\frac{\partial h}{\partial s} \right)_{p,q}, \quad A = -\left(\frac{\partial h}{\partial q} \right)_{p,s} \tag{10}$$

Consider the case in which the wave front is an outgoing characteristic. Then the boundary conditions to the system can be written in terms of α and ψ as

$$u = \frac{\mathrm{d}}{\mathrm{d}\alpha} F(\alpha), \; x = F(\alpha), \; t = \alpha \text{ at } \psi = 0 \tag{11}$$

$$p = p_0, \; s = s_0, \; q = q_0, \; t = \psi \text{ at } \alpha = 0 \tag{12}$$

where $x = F(t)$ denotes the piston path in the x-t plane and the subscript 0 denotes an undisturbed gas.

It is easy to see that the form of the system of equations, Eqs. (1-4) and Eqs. (7-10) remain unchanged if the time t is nondimensionalized by an appropriate characteristic time τ^*, the distance x by $c_{f0}\tau^*$, the density ρ by ρ_0, the velocity u by c_{f0}, the pressure p by $\rho_0 c_{f0}^2$, the progress variable q by q_0, the rate of internal transformation Q by q_0/τ^*, the temperature T by T_0, the entropy s by c_{f0}^2/T_0, the affinity A by c_{f0}^2/q_0, the enthalpy h by c_{f0}^2, and the frozen speed of sound c_f by c_{f0}. Also the form of the first of the boundary conditions, Eq. (11), is unchanged if the coordinate of the piston path $F(t)$ is nondimensionalized by $c_{f0}\tau^*$ and the piston velocity $\mathrm{d}F(t)/\mathrm{d}t$ by c_{f0}. It is, therefore, unnecessary to introduce new notations to denote the dimensionless variables. Henceforth, all variables appearing in the system of basic equations and the boundary conditions (11) can be regarded as dimensionless. Only the conditions (12) must be rewritten in the nondimensional form as follows

$$p = p_0, \quad s = s_0, \quad q = 1, \quad t = \psi \text{ at } \alpha = 0 \tag{13}$$

It must be noticed that in this case the characteristic variables α and ψ are dimensionless to begin with.

Considering the case of

$$F(t) = \epsilon f(t) \tag{14}$$

where

$$\epsilon \ll 1, \quad f(t) = O(1) \tag{15}$$

we can assume a solution to the problem of the form

$$\begin{aligned} &\rho = 1 + \epsilon\rho^{(1)} + O(\epsilon^2), \quad u = \epsilon u^{(1)} + O(\epsilon^2) \\ &p = p_0 + \epsilon p^{(1)} + O(\epsilon^2), \quad q = 1 + \epsilon q^{(1)} + O(\epsilon^2) \\ &s = s_0 + \epsilon s^{(1)} + O(\epsilon^2), \quad c_f = 1 + \epsilon c_f^{(1)} + O(\epsilon^2) \\ &x = \psi + \epsilon x^{(1)} + O(\epsilon^2), \quad t = \alpha + \psi + \epsilon t^{(1)} + O(\epsilon^2) \end{aligned} \tag{16}$$

Substituting these into the system of basic equations and collecting terms of the order of ϵ, we obtain the system of equations for the first-order problem

$$\begin{aligned} &p_\alpha^{(1)} - u_\alpha^{(1)} - p_\psi^{(1)} = 0, \quad p_\psi^{(1)} + u_\psi^{(1)} = \mathrm{d}/\tau_0 (q^{(1)} - cp^{(1)}) \\ &q_\alpha^{(1)} = -1/\tau_0 (q^{(1)} - cp^{(1)}), \quad s_\alpha^{(1)} = 0 \\ &c_f^{(1)} = ap^{(1)} - bq^{(1)} \\ &x_\alpha^{(1)} = u^{(1)}, \qquad x_\psi^{(1)} - t_\psi^{(1)} = u^{(1)} + c_f^{(1)} \end{aligned} \tag{17}$$

where the parameters a, b, c, d, and τ_0 are constants defined by

$$a = \left(\frac{\partial c_f}{\partial p} \right)_{s,q,o}, \quad b = -\left(\frac{\partial c_f}{\partial q} \right)_{p,s,o}, \quad c = \left(\frac{\partial \bar{q}}{\partial p} \right)_{s,o}$$

$$d = \left(\frac{\partial \rho}{\partial q} \right)_{p,s,o}, \quad \tau_0 = -1/\left(\frac{\partial Q}{\partial q} \right)_{p,s,o} \tag{18}$$

and $\bar{q} = \bar{q}(p,s)$ is the equilibrium value of q and can be obtained from $Q = 0$ or $A = 0$. Similarly from Eqs. (11) and (12), the boundary conditions to be applied to the first-order problem can be obtained as follows

$$u^{(1)} = f'(\alpha), \; x^{(1)} = f(\alpha), \; t^{(1)} = 0 \quad \text{at} \quad \psi = 0 \tag{19}$$

$$u^{(1)} = p^{(1)} = q^{(1)} = s^{(1)} = x^{(1)} = t^{(1)} = 0 \quad \text{at} \quad \alpha = 0 \tag{20}$$

Now in order to specify the gas process and to determine the constants defined by Eq. (18), the equations for $h(p,s,q)$ and $Q(p,s,q)$ must be known. For a vibrationally relaxing diatomic gas, these are given in the nondimensional form as follows

$$h = \frac{5}{2}\frac{\xi_0}{\xi} + \frac{5}{7}\frac{\xi_0}{\exp\xi_0 - 1} q \tag{21}$$

$$Q = -\frac{1}{\tau(p,s,q)} (q - q_e) \tag{22}$$

where

$$q = \frac{\exp\xi_0 - 1}{\exp \xi_V - 1}, \quad q_e = \frac{\exp\xi_0 - 1}{\exp \xi - 1} \tag{23}$$

$$\xi_V = \frac{\theta}{T_V}, \quad \xi = \frac{\theta}{T}, \quad (\xi_0 = \theta) \tag{24}$$

and where q is taken to be the vibrational energy, and T_V is the vibrational temperature, θ the characteristic temperature of vibration, which are both nondimensionalized by the temperature of the undisturbed gas. The inverse temperature ξ is given as a function of p, $q(\xi_V)$ and s as follows,

$$\frac{\xi}{\xi_0} = \left\{ \frac{p_0}{p} \frac{\exp\xi_0 - 1}{\exp \xi_V - 1} \exp\left(\frac{\xi_V \exp\xi_V}{\exp \xi_V - 1} - \frac{\xi_0 \exp\xi_0}{\exp \xi_0 - 1} \right) \exp\left[-\frac{7}{5}(s - s_0) \right] \right\}^{2/7} \tag{25}$$

It is obvious that $\bar{q}=q(\bar{\xi})=q_e(\bar{\xi})$, where $\bar{\xi}$ is determined from this equation by letting $\xi_V=\xi=\bar{\xi}$. It is not necessary for our present purpose to specify the equation for the relaxation time $\tau(p,s,q)$.†

III. Method of Integral Relations

It is sufficient for our purpose to consider the solution of this system of equations in a domain $\alpha\geq 0$ and $\psi\geq 0$, which corresponds to a domain $x\geq F(t)$ and $x\leq t(t\geq 0)$ in the x-t plane. These are schematically shown in Fig. 1 as shaded regions for a typical case. The flow domain in the ψ-α plane is first divided into a large number of strips by drawing equidistant lines on $\alpha=i\Delta\alpha$, where i is a nonnegative integer, and $\Delta\alpha$ is a positive constant specified properly. Next suppose that we approximate all the variables in the system by linear interpolation functions of α in a domain $i\Delta\alpha\leq\alpha\leq(i+1)\Delta\alpha$ and $\psi\geq 0$,

$$P(\alpha,\psi)=P_i(\psi)+\frac{(\alpha-i\Delta\alpha)}{\Delta\alpha}\{P_{i+1}(\psi)-P_i(\psi)\} \tag{26}$$

where

$$P_i(\psi)=P(i\Delta\alpha,\psi) \tag{27}$$

and P may denote any one of the dependent variables u, p, q, s, t, and x.[9] By substituting relations of the form of Eq. (26) into the system of basic equations (17), and integrating the equations along an arbitrary line $\psi=$const. from $\alpha=i\Delta\alpha$ to $\alpha=(i+1)\Delta\alpha$, we obtain a system of equations

$$\frac{dp_{i+1}^{(1)}}{d\psi}-\frac{2}{\Delta\alpha}p_{i+1}^{(1)}+\frac{2}{\Delta\alpha}u_{i+1}^{(1)}=-\frac{dp_i^{(1)}}{d\psi}-\frac{2}{\Delta\alpha}p_i^{(1)}+\frac{2}{\Delta\alpha}u_i^{(1)}$$

$$\frac{dp_{i+1}^{(1)}}{d\psi}+\frac{du_{i+1}^{(1)}}{d\psi}+\frac{2m}{2\tau_0+\Delta\alpha}p_{i+1}^{(1)}$$

$$=-\frac{dp_i^{(1)}}{d\psi}-\frac{du_i^{(1)}}{d\psi}-\frac{2m}{2\tau_0+\Delta\alpha}p_i^{(1)}+\frac{4d}{2\tau_0+\Delta\alpha}q_i^{(1)}$$

$$q_{i+1}^{(1)}-\frac{c\Delta\alpha}{2\tau_0+\Delta\alpha}p_{i+1}^{(1)}=\frac{c\Delta\alpha}{2\tau_0+\Delta\alpha}p_i^{(1)}+\frac{2\tau_0-\Delta\alpha}{2\tau_0+\Delta\alpha}q_i^{(1)}$$

$$s_{i+1}^{(1)}=s_i^{(1)}$$

$$c_{fi+1}^{(1)}-ap_{i+1}^{(1)}+bq_{i+1}^{(1)}=0$$

$$x_{i+1}^{(1)}-\frac{1}{2}\Delta\alpha u_{i+1}^{(1)}=x_i^{(1)}+\frac{1}{2}\Delta\alpha u_i^{(1)}$$

$$\frac{dt_{i+1}^{(1)}}{d\psi}-\frac{dx_{i+1}^{(1)}}{d\psi}+u_{i+1}^{(1)}+c_{fi+1}^{(1)}=-\frac{dt_i^{(1)}}{d\psi}+\frac{dx_i^{(1)}}{d\psi}-u_i^{(1)}-c_{fi}^{(1)} \tag{28}$$

where

$$m=cd \tag{29}$$

In the case of the characteristic wave front, the boundary conditions (19) and (20) must be replaced by

$$u_i^{(1)}(0)=u^{(1)}(i\Delta\alpha,0)=f'(i\Delta\alpha),\quad t_i^{(1)}(0)=0 \tag{30}$$

$$u_0^{(1)}(\psi)=p_0^{(1)}(\psi)=s_0^{(1)}(\psi)=q_0^{(1)}(\psi)=x_0^{(1)}(\psi)$$

$$=t_0^{(1)}(\psi)=0 \tag{31}$$

†Note that $\tau_0=-1/(\partial Q/\partial q)_{p,s,0}=\tau(p_0,s_0,q_0)/[1-(dq_e/d\xi)_0(\partial\xi/\partial q)_{p,s,0}]$

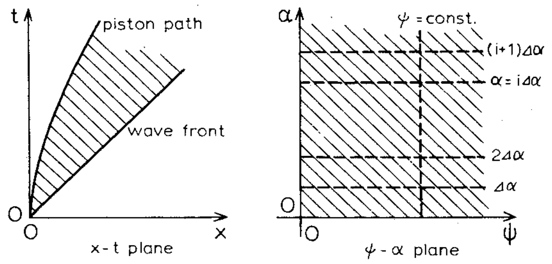

Fig. 1 Flowfield.

Furthermore in order to warrant a continuous solution to the system, the following conditions must be added to the previous conditions

$$P_i(\psi)\rightarrow P_0(\psi)\quad\text{as}\quad\Delta\alpha\rightarrow 0,$$

$$(P=u^{(1)},p^{(1)},q^{(1)},s^{(1)},x^{(1)},t^{(1)}) \tag{32}$$

which are also necessary for singling out a unique flow pattern.

It is easy to see that the solution to the system satisfying the conditions (30-32) has the following form

$$u_i^{(1)}=e^{\lambda\psi}\sum_{j=0}^{i-1}U_{ij}\psi^j,\quad p_i^{(1)}=e^{\lambda\psi}\sum_{j=0}^{i-1}P_{ij}\psi^j$$

$$q_i^{(1)}=e^{\lambda\psi}\sum_{j=0}^{i-1}Q_{ij}\psi^j,\quad S_i^{(1)}=0$$

$$x_i^{(1)}=e^{\lambda\psi}\sum_{j=0}^{i-1}X_{ij}\psi^j,\quad t_i^{(1)}=e^{\lambda\psi}\sum_{j=0}^{i-1}T_{ij}\psi^j-T_{i0} \tag{33}$$

where

$$\lambda=\frac{2}{\Delta\alpha}\left\{1-\sqrt{1+\frac{m\Delta\alpha}{2\tau_0+\Delta\alpha}}\right\} \tag{34}$$

and U_{ij}, P_{ij}, Q_{ij}, X_{ij}, and T_{ij} are constants independent of ψ. The system of equations (28) may be thought of as relating $(u_i^{(1)},p_i^{(1)},q_i^{(1)},s_i^{(1)},x_i^{(1)},t_i^{(1)})$ to $(u_{i+1}^{(1)},p_{i+1}^{(1)},q_{i+1}^{(1)},s_{i+1}^{(1)},x_{i+1}^{(1)},t_{i+1}^{(1)})$. Substituting Eq. (33) into Eq. (28), we can determine the distributions of flow variables $(u_{i+1}^{(1)},p_{i+1}^{(1)},q_{i+1}^{(1)},s_{i+1}^{(1)},x_{i+1}^{(1)},t_{i+1}^{(1)})$ with the conditions of Eqs. (30) and (31) by only straightforward algebraic manipulation.

On the line $\alpha=0$ (for $i=0$), all the variables are equal to those of undisturbed gas and are known. Once these are known, the solution to Eq. (28) yields the values of the flow

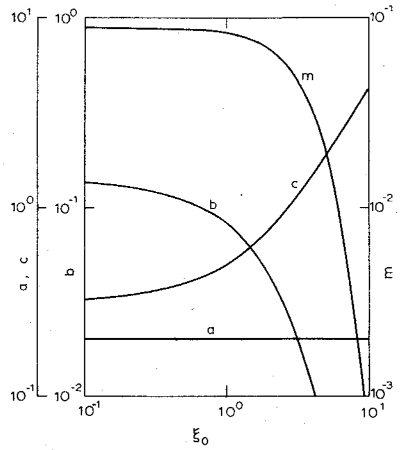

Fig. 2 Parameters *a*, *b*, *c*, and *m* for a vibrationally relaxing diatomic gas.

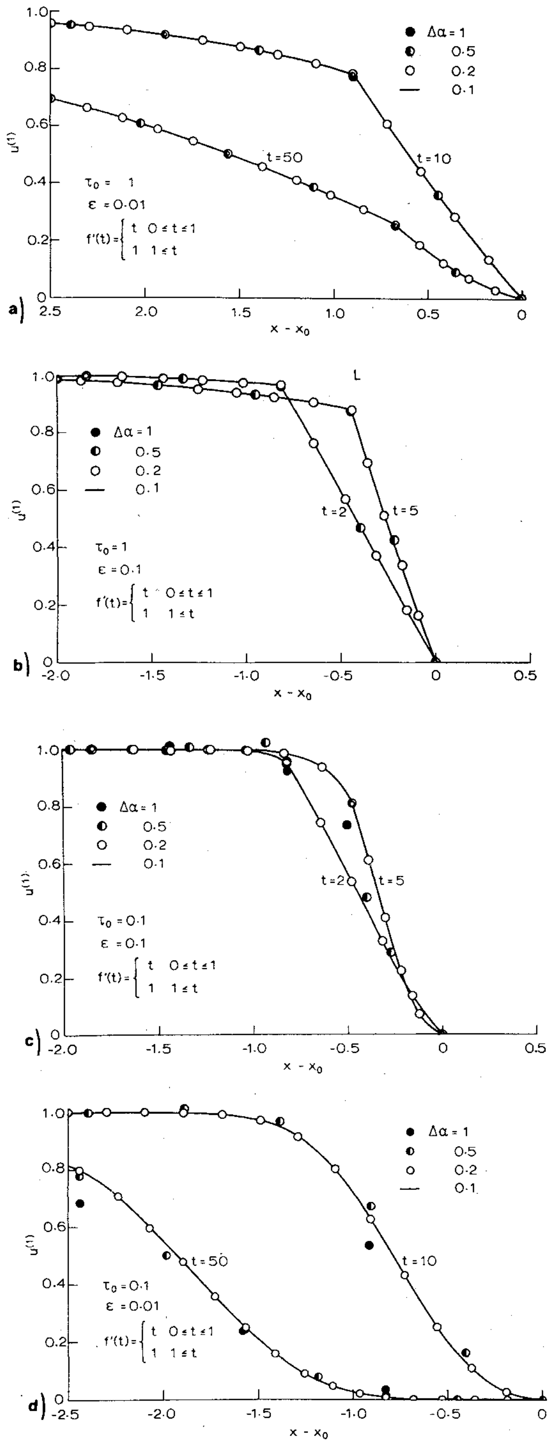

Fig. 3 Velocity distributions in a wave calculated for $\Delta\alpha = 0.1$, 0.2, 0.5, and 1.0.

variables on the line $\alpha = \Delta\alpha$ $(i=1)$. By repeated applications of this procedure, the flow properties on the line $\alpha = i\Delta\alpha$ (for any value of i) may be uniquely determined. The possibility of such step-by-step determination of the flowfield arises from the hyperbolic nature of the system of basic equations and the use of the characteristics as the independent variables.

IV. Numerical Results

To check the results and investigate the real gas effects on the wave phenomena, sample numerical calculations were systematically carried out for a vibrationally relaxing diatomic gas on the electronic digital computer. The parameters a, b, c, and m $(=cd)$ defined by Eq. (18) were first calculated and the results are shown in Fig. 2. These have been used in calculating the flowfields.

Four types of piston velocities are considered

$$\text{(a) } f'(t) = \begin{cases} t & 0 \le t \le 1 \\ 1 & 1 \le t, \end{cases} \qquad \text{(b) } f'(t) = \begin{cases} -t & 0 \le t \le 1 \\ -1 & 1 \le t \end{cases}$$

$$\text{(c) } f'(t) = \begin{cases} t & 0 \le t \le 1 \\ 2-t & 1 \le t \le 2 \\ 0 & 2 \le t, \end{cases} \qquad \text{(d) } f'(t) = \begin{cases} -t & 0 \le t \le 1 \\ t-2 & 1 \le t \le 2 \\ 0 & 2 \le t \end{cases} \tag{35}$$

The velocity distributions in waves calculated for these types of piston velocities are shown in Figs. 3–9. A disadvantage of the scheme employed here lies in the fact that the results of the method depend upon the choice of the magnitude of $\Delta\alpha$ and the form of interpolation functions. Here only one form of interpolation functions, Eq. (26), was used but the numerical check on the results has been made for several values of $\Delta\alpha$. Figs. (3a–3d) show velocity distributions in nonlinear waves produced by a piston with velocity (a). Four cases are considered; $(\epsilon, \tau_0) = (0.1, 1)$, $(0.1, 0.1)$, $(0.01, 1)$, and $(0.01, 0.1)$. In each case four calculations were made for $\Delta\alpha = 0.1$, 0.2, 0.5, and 1, respectively, with the same condition $\xi_0 = 1$. From these figures we can easily see the satisfactory convergence of results with decreasing $\Delta\alpha$. Closer investigation has made it clear that the accuracy of the results mainly depends upon the magnitude of the ratio $\Delta\alpha/\tau_0$ instead of the value of $\Delta\alpha$ itself. It may well be concluded that the result is practically taken to be exact if the selected value of the ratio $\Delta\alpha/\tau_0$ does not much exceed unity.

In the calculations shown in Figs. 4–9, the strip size $\Delta\alpha$ is taken to be 0.2, which may be thought of as being small

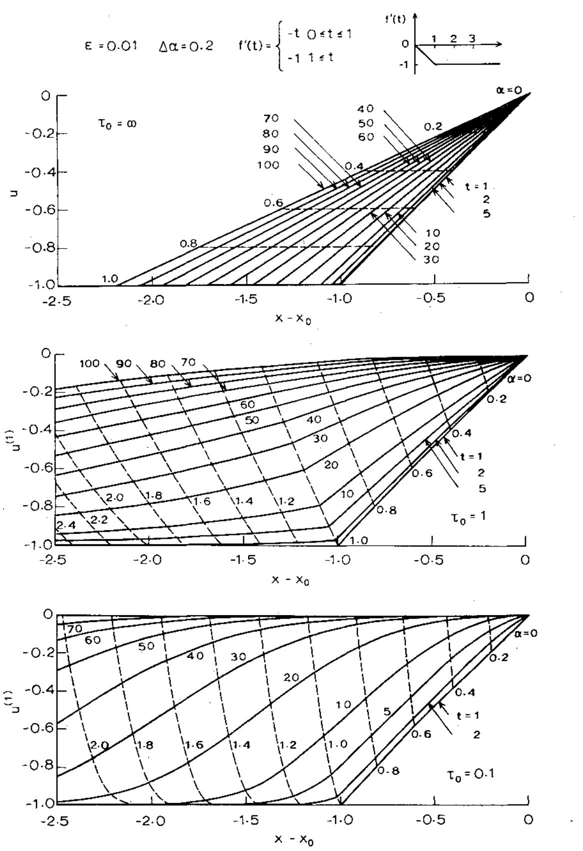

Fig. 4 Velocity distributions in a wave.

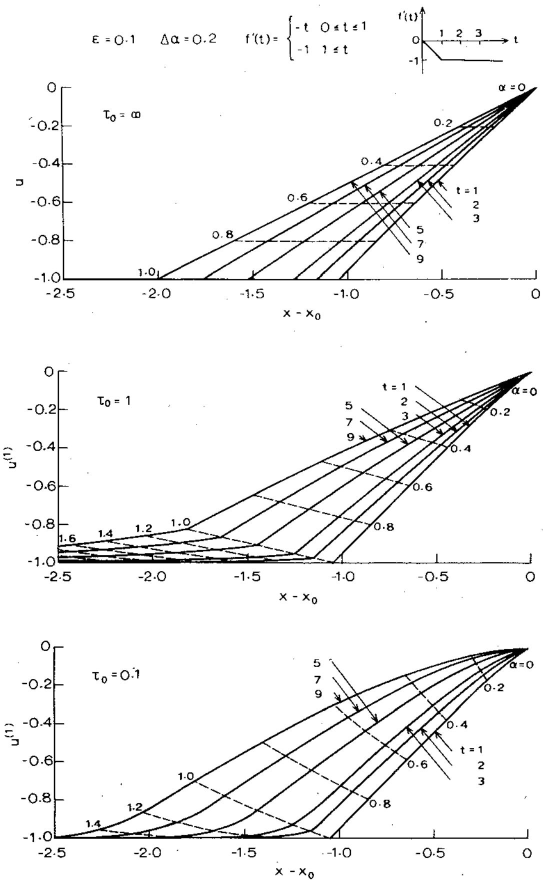

Fig. 5 Velocity distributions in a wave.

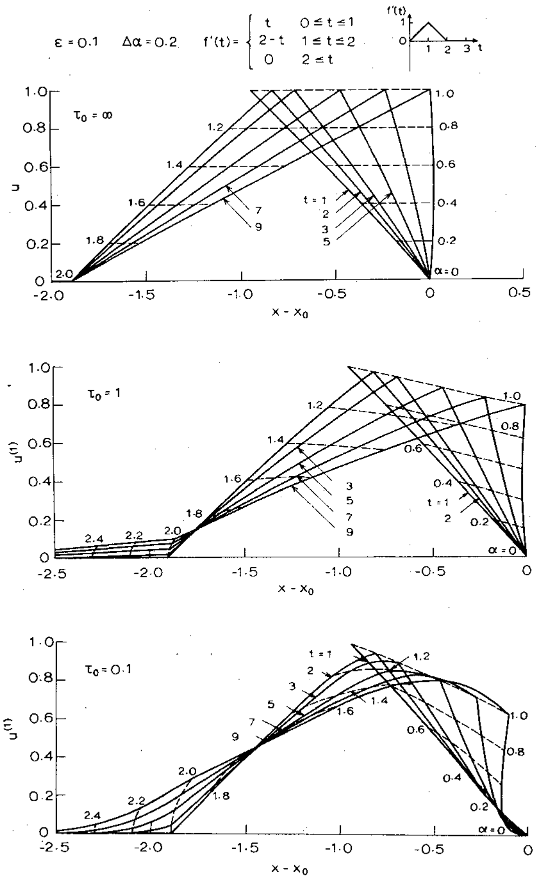

Fig. 7 Velocity distributions in a wave.

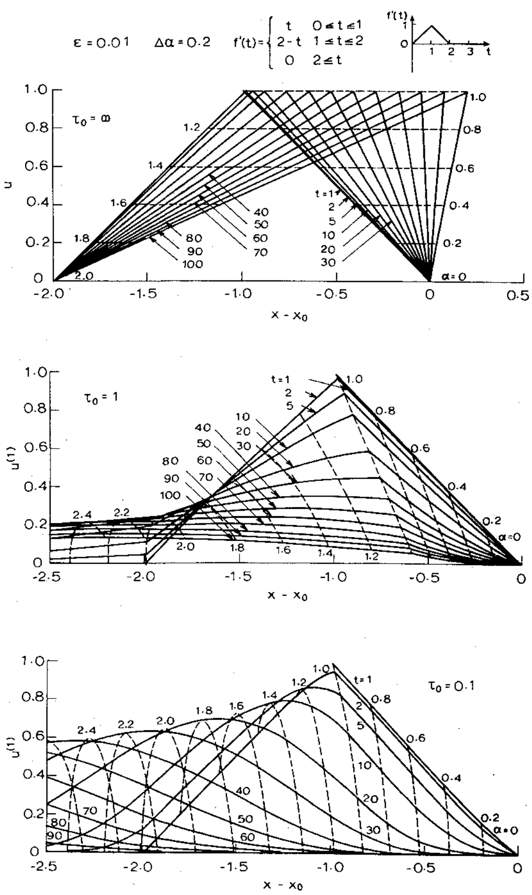

Fig. 6 Velocity distributions in a wave.

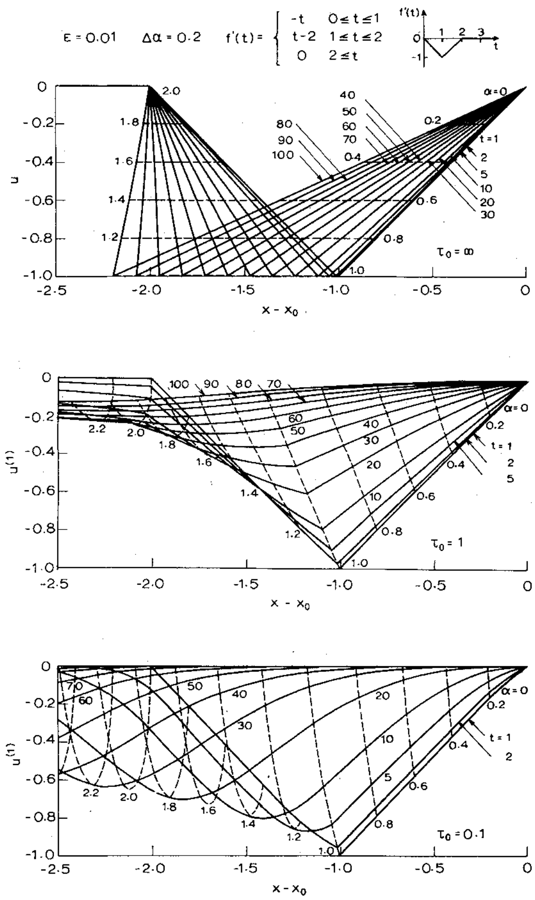

Fig. 8 Velocity distributions in a wave.

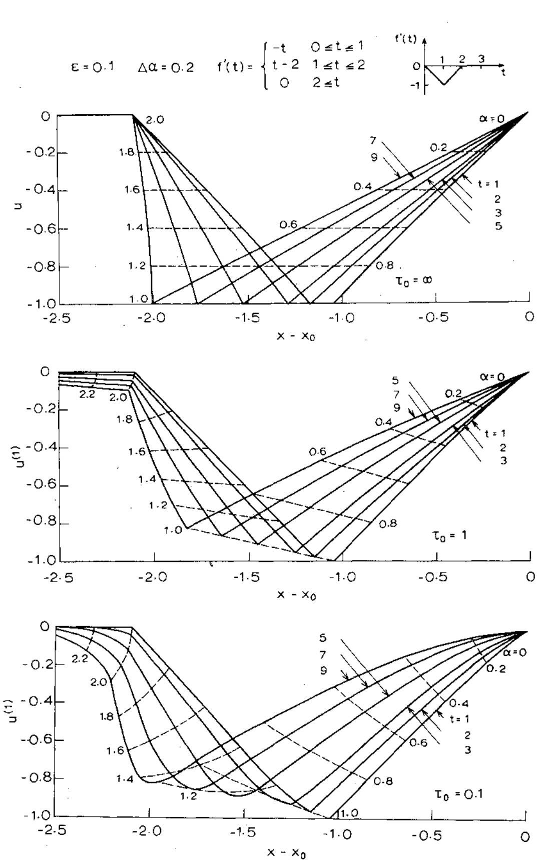

Fig. 9 Velocity distributions in a wave.

enough for sufficiently exact calculations from the numerical check previously given. The parameter ξ_0 is chosen as unity, for which the other parameters, *a, b, c,* and *m* are given as follows

$$a=0.2,\ b=0.08314,\ c=0.50101,\ m=0.08331 \tag{36}$$

See, for example, Fig. 4. The origin of the abscissa is taken to be at $x=x_0$, where $x_0=x_0(t)$ denotes the coordinate of the wave front in the *x-t* plane. Velocity distributions in a wave are shown as solid lines along the distance from the wave front, $(x-x_0)$, at various nondimensional times, which are measured from the time when the piston is set in motion. For one form of the piston velocity, three values of τ_0 are considered. The upper group of curves show the velocity distributions for $\tau_0=\infty$, the middle group for $\tau_0=1$, and the lower group for $\tau_0=0.1$. In each group the loci of the positive characteristics are shown as dotted lines. Along each dotted line the value of α is constant at the value specified.

Since for the frozen flows the problem becomes the same one as for the flows of a classical ideal gas and can be solved exactly, the results for $\tau_0=\infty$ in each figure are not those obtained by the approximate method developed here but those calculated from the exact solution. The self-steepening tendencies in the compression phase of waves are clearly seen for the frozen flows in Figs. 6-9. It must be noticed that after the shock is formed ($t_\alpha=0$) the characteristic wave front must be replaced by a shock front. Here, however, the wave phenomena after the formation of the shock are not treated, and therefore the shocks have not been introduced in the figures even after $t_\alpha=0$.

It is important to notice that Figs. 3 and 6–9 show well that the self-steepening tendencies in the compression phase (due to the nonlinear effects) may be appreciably resisted by the relaxation effects alone without assistance from viscosity and conduction. When the relaxation effects are greater than the nonlinear effects, the velocity gradient even in the compression phase is gradually weakened in the course of propagation, which does not happen in classical gas dynamics. Such a case is shown in Figs. 3, 6, and 8. For large time or for large ψ the wave form near the front is decisively affected by the conflicting effects of relaxation and convection.

It is of interest to investigate the behavior of weak discontinuities in derivatives of flow variables. The piston velocities considered here have a few acceleration discontinuities and give the same kind of discontinuities to the gas adjacent to the piston surface, which are propagated with waves along particular positive characteristics. In the relaxing gas, these decay in the course of propagation and finally disappear in the limit $t\to\infty$ or $\psi\to\infty$. The decay rates of such discontinuities are the greater for the smaller values of τ_0 (more strictly τ_0/m). It is possible to give more exact discussions of the propagation of weak discontinuities. Let's assume that the flow variables themselves and their derivatives with respect to ψ are continuous everywhere in the flowfield and only the discontinuities in derivatives with respect to α can exist, and furthermore assume that these discontinuities can be propagated with waves only along particular positive characteristics $\alpha=$ const. If along $\alpha=\alpha_c$ the weak discontinuities are propagated with a wave, we get from the basic equations (17)

$$[u_\alpha^{(1)}]_c=[p_\alpha^{(1)}]_c=[f''(\alpha)]_c e^{-(m/2\tau_0)\psi}$$

$$[q_\alpha^{(1)}]_c=[s_\alpha^{(1)}]_c=[x_\alpha^{(1)}]_c=0$$

$$[t_\alpha^{(1)}]_c=2\frac{\tau_0}{m}(1+a)[f''(\alpha)]_c(e^{-(m/2\tau_0)\psi}-1) \tag{37}$$

where $[\ \ ]_c$ denotes the difference of derivatives of a flow variable just ahead of and behind the characteristic $\alpha=\alpha_c$. Since to the first approximation

$$\partial u/\partial x \sim u_\psi^{(1)}-u_\alpha^{(1)}$$

we have

$$\left[\frac{\partial u}{\partial x}\right]_c \sim -[u_\alpha^{(1)}]_c=-[f''(\alpha)]_c\, e^{-(m/2\tau_0)\psi} \tag{38}$$

which explains both qualitatively and quantitatively the numerical results shown in the figures.

From the behaviors of the loci of the positive characteristics we can easily see how the waves are convected with fluid, which is moving in the direction of propagation in the compression phase and in the opposite direction in the expansion phase. It is interesting that in the case of piston velocity (c) the curves for the loci of characteristics in the rear parts of waves have the maximum points (Fig. 6) and on the contrary in the case of (d) these have the minimum points (Fig. 8).

The waves are also distorted partly by frequency dispersion due to the vibrational relaxation, which is apparently divided into two parts, diffusion and attenuation of waves, and is clearly manifesting itself in the figures. The diffusion of waves are prominent especially in Figs. 6-9. It is of interest that in the early stage of propagation the diffusivity of waves for $\tau_0=1$ is more distinguished than that for $\tau_0=0.1$. The attenuation of waves is greatest near the wavefront at least in the first-order problems. Though for the classical ideal gas the rate of steepening of wave-forms is greatest at the front for the velocity forms of (a) and (c), the same cannot be said for the relaxing gas. Therefore it may happen that a shock is not first formed at the characteristic front but is first formed somewhat behind the front.

V. Concluding Remarks

The method of integral relations has been applied to the problem of solving one-dimensional nonsteady flows of an inviscid relaxing gas. An approximate semi-analytical solution was constructed for the first-order problem in the characteristic plane. From the numerical check by the sample calculations, it may be concluded that our solution is quite accurate and reliable at least under the conditions for the magnitude of the ratio $\Delta\alpha/\tau_0$ and the forms of piston velocities considered here. It must be emphasized that our scheme has very important advantages for being well-suited for automatic digital computation and for treating the problem with discontinuities in derivatives of flow variables, and for requiring very little computing time.

Though the change in energy associated with the vibrational mode forms little part of the total change of enthalpy of the gas, the effects of the vibrational relaxation on the wave forms are cumulative and can therefore become very significant in the course of propagation. When the shock is formed from the simple positive characteristics during the propagation of a wave, the problem becomes considerably difficult to treat. However in the case of the shock front which is produced by a piston when it is suddenly set in motion into the cylinder, the scheme developed here remains valid if the appropriate shock conditions are introduced instead of the boundary conditions (31).[8] Numerical calculations in this case can be carried out in almost the same manner as in the case of the characteristic front. The results have not been presented here for lack of space, but will be reported elsewhere in the near future.

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